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Spin down problem of rotating stratified fluid in thermally insulated circular cylinders

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A response of viscous heat-conducting compressible fluid to an abrupt change of angular velocity of a containing thermally insulated circular cylinder under the existence of stable distribution of the temperature is investigated within the framework of the Boussinesq approximation for a time duration of the order of the homogeneous-fluid spin down time in order to resolve the Holton–Pedlosky controversy. The explicit expression of the solution is obtained by the standard method and Holton's conclusion is confirmed. The secondary meridional current induced by the Ekman layers spins the fluid down to a quasi-steady state within the present time scale. However, unlike the homogeneous case, the quasi-steady state is not one of solid body rotation. The final approach to the state of rigid rotation is achieved via the viscous diffusion in the time scale of the usual diffusion time.

1. Introduction

The spin down problem of rotating stratified fluid is of central interest to some astronomers and hydrodynamicists.

The astronomical motivation comes from Dicke's hypothesis that the solar interior rotates much faster than the solar surface (Dicke 1964). Howard, Moore & Spiegel (1967) pointed out the existence of the solar spin down process similar to that in the rotating incompressible fluid and opposed Dicke's hypothesis. Dicke denied the existence of this spin down procedure on the basis of the steady solution in the solar interior and also on Pedlosky's (1967) results, and insisted on the validity of his hypothesis (Dicke 1967).

The direct hydrodynamical motivation results from the controversy between Holton's and Pedlosky's results about the case with a thermally insulated boundary (not to mention the above fundamental motivation). Holton investigated the spin down of rotating stratified fluid in a circular cylinder and showd the existence of a spin down procedure similar, but not completely the same, to that in homogeneous fluids (Holton 1965). Pedlosky re-examined the same problem from this view point to complete Holton's heuristic treatment and gave a clearer formulation. His conclusion for the case with a thermally insulated boundary is that the interior region is spun down by a strictly diffusive process within the time scale of the diffusion time, which conclusion is completely at variance with that of Holton (Pedlosky 1967). Holton & Stone (1968) examined

Pedlosky's solution, pointed out the inconsistency of the solution, and requested the further investigation of the same problem. The purpose of this paper is to resolve this controversy.

Viscous heat conducting compressible fluid fills, and rotates with, a circular cylinder rotating with constant angular velocity around the vertical axis of symmetry. The temperature near the top is held higher than that near the bottom to establish a stable distribution of the temperature. Then, the angular velocity of the cylinder is changed abruptly after a certain instant while the boundary is made thermally insulated. Our problem is to investigate the response of the fluid to this abrupt change for the time duration of the homogeneous-fluid spin down time within the framework of the Boussinesq approximation.

The author also investigated the case with given boundary temperature and gave an explicit solution which is in qualitative agreement with that of Holton (Sakurai 1969). The present paper is an extension of his previous results to the case with a thermally insulated boundary.

2. Basic equations

The basic equations governing the axisymmetric motion of viscous heat conducting compressible fluid within the framework of the Boussinesq approximation written in cylindrical co-ordinates rotating with the angular velocity Ω about the vertical axis of symmetry are:

$$\frac{\partial q_r}{\partial t} - 2q_\theta = -\frac{\partial p}{\partial r} + E\mathscr{L}q_r, \tag{1}$$

$$\frac{\partial q_\theta}{\partial t} + 2q_r = E\mathscr{L}q_\theta, \tag{2}$$

$$\frac{\partial q_z}{\partial t} = -\frac{\partial p}{\partial z} + E\Delta q_z + T, \tag{3}$$

$$\sigma\left(\frac{\partial T}{\partial t} + Sq_z\right) = E\Delta T, \tag{4}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(rq_r) + \frac{\partial q_z}{\partial z}, \tag{5}$$

$$E = \frac{\nu}{L^2\Omega}, \quad S = \frac{\alpha g(\overline{T}_1 - \overline{T}_0)}{L\Omega^2}, \quad \sigma = \frac{\nu}{\kappa}, \tag{6}$$

$$\mathscr{L} = \Delta - \frac{1}{r^2}, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \tag{7}$$

where

$$(\bar{r}, \theta, \bar{z}) = (Lr, \theta, Lz), \quad \bar{t} = t/\Omega, \tag{8}$$

$$\bar{\mathbf{q}} = \epsilon\Omega L\mathbf{q}, \quad \overline{p} = \overline{p}_s + \epsilon\Omega^2L^2\overline{\rho}_s p, \quad \overline{T} = \overline{T}_s + \frac{\epsilon L\Omega^2}{\alpha g}T, \tag{9}$$

$$\overline{\rho}_s = \overline{\rho}_0\left\{1 - \frac{\alpha(\overline{T}_1 - \overline{T}_0)\bar{z}}{L}\right\}, \quad \overline{p}_s = \overline{p}_0 - \int_0^{\bar{z}} g\overline{\rho}_s d\bar{z}, \quad \overline{T}_s = \overline{T}_0 + \frac{\overline{T}_1 - \overline{T}_0}{L}\bar{z}, \tag{10}$$

and (q_r, q_θ, q_z), p, ρ, T, (r, θ, z), t, ν, κ, α, g, L and ϵ are respectively the velocity, the pressure, the density, the temperature, the position vector, the time, the viscosity, the thermal conductivity, the coefficient of thermal expansion, the gravitational acceleration, the height of the cylinder and the parameter corresponding to the small deviation from the state of rotating equilibrium. Suffixes s, 1 and 0 and bars on letters refer to the state of rotating equilibrium, the top and bottom of the cylinder and to the original physical quantity with dimension, respectively. In the derivation of the above basic equations from those of the Boussinesq approximation, terms of the order of ϵ^2 and of $L\Omega^2/g$ are neglected, as is exemplified for the steady flow (Barcilon & Pedlosky 1967). E is assumed to be very small while S and σ are taken to be of the order of 1, hereafter.

The introduction of the stream function of the meridional current leads to the following:

$$\mathrm{E}\tilde{\mathscr{L}}\left\{E^{\frac{1}{2}}\tilde{\mathscr{L}}-\frac{\partial}{\partial\tilde{t}}\right\}\tilde{\psi}+2\frac{\partial\tilde{q}_\theta}{\partial\tilde{z}}=\frac{\partial\tilde{T}}{\partial\tilde{r}}, \tag{11}$$

$$\left\{E^{\frac{1}{2}}\tilde{\mathscr{L}}-\frac{\partial}{\partial\tilde{t}}\right\}\tilde{q}_\theta-2\frac{\partial\tilde{\psi}}{\partial\tilde{z}}=0, \tag{12}$$

$$\left\{E^{\frac{1}{2}}\tilde{\Delta}-\sigma\frac{\partial}{\partial\tilde{t}}\right\}\tilde{T}+\frac{\sigma S}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}(\tilde{r}\tilde{\psi})=0, \tag{13}$$

where
$$q_r=\frac{\partial\psi}{\partial z},\quad q_z=-\frac{1}{r}\frac{\partial}{\partial r}(r\psi), \tag{14}$$

$$\left.\begin{aligned} r=\tilde{r},\quad z=\tilde{z},\quad t=E^{-\frac{1}{2}}\tilde{t},\\ \psi=E^{\frac{1}{2}}\tilde{\psi},\quad q_\theta=\tilde{q}_\theta,\quad T=\tilde{T}.\end{aligned}\right\} \tag{15}$$

The transformation (15) corresponds to the time duration in which our investigation is aimed. The quantities with tildes are assumed to be of the order of 1. Hence, equations (15) imply the possible existence of the meridional current with the same order of magnitude as that of the rotating homogeneous fluid.

The initial conditions are obtained by the expression that the fluid is in rotating equilibrium until a certain instant:

$$\tilde{\psi}=\tilde{q}_\theta=\tilde{T}=0\quad\text{for}\quad 0\leqslant\tilde{z}\leqslant 1,\quad 0\leqslant\tilde{r}\leqslant r_0,\quad \tilde{t}\leqslant 0, \tag{16}$$

where r_0 is the ratio of the radius to the height of the cylinder.

The boundary conditions are obtained by expression of the fact that the angular velocity of the cylinder is changed abruptly after the above instant while the boundary is made thermally insulated:

$$\tilde{\psi}=\frac{\partial\tilde{\psi}}{\partial\tilde{z}}=\frac{\partial\tilde{T}}{\partial\tilde{z}}=0,\quad \tilde{q}_\theta=\omega\tilde{r}\quad\text{for}\quad\tilde{z}=0\quad\text{or}\quad 1,\quad 0\leqslant\tilde{r}\leqslant r_0,\quad\tilde{t}>0. \tag{17}$$

$$\tilde{\psi}=\frac{\partial\tilde{\psi}}{\partial\tilde{r}}=\frac{\partial\tilde{T}}{\partial\tilde{r}}=0,\quad \tilde{q}_\theta=\omega r_0\quad\text{for}\quad 0\leqslant\tilde{z}\leqslant 1,\quad\tilde{r}=r_0,\quad\tilde{t}>0. \tag{18}$$

The tildes over letters are omitted, hereafter, for the sake of simplicity.

3. Similar solution in the limiting case of infinitely large radius of the cylinder

Before going directly into the solution of our problem, it is interesting to note that there is the following similarity solution for the limiting case with infinitely large radius:

$$\psi = r\phi(z,t), \quad q_\theta = rV(z,t), \quad T = T(z,t). \tag{19}$$

The substitution of (19) into (11) to (13) leads to the following:

$$E\frac{\partial^2}{\partial z^2}\left\{E^{\frac{1}{2}}\frac{\partial^2}{\partial z^2}-\frac{\partial}{\partial t}\right\}\phi+2\frac{\partial V}{\partial z}=0, \tag{20}$$

$$\left\{E^{\frac{1}{2}}\frac{\partial^2}{\partial z^2}-\frac{\partial}{\partial t}\right\}V-2\frac{\partial \phi}{\partial z}=0, \tag{21}$$

$$\left\{E^{\frac{1}{2}}\frac{\partial^2}{\partial z^2}-\sigma\frac{\partial}{\partial t}\right\}T+2\sigma S\phi=0. \tag{22}$$

Equations (20) and (21) are those already treated for the homogeneous case (Greenspan & Howard 1963). Therefore, it is concluded that the state of rigid rotation is established within the time scale of incompressible spin down time without regard to the stratification in the present limiting case. An important point to be stressed here is that this limiting case is not simply isolated from other cases but is tended continuously to as r_0 is made larger and larger. Pedlosky's solution (equation (6.37) in his paper) apparently violates this condition, and cannot be a correct one. Another interesting phenomenon which the above similarity solution implies is that the thickness of the thermal boundary layer is of the order of $E^{\frac{1}{4}}$, as is easily seen from (22). The meridional current driven by the Ekman layer modifies not only the velocity but also the temperature outside the boundary layer. The matching of this modified temperature with that of the boundary is achieved via thermal boundary layer of the above thickness. The appearance of this horizontal intermediate layer is characteristic of the transient variation within the time scale of the incompressible spin down time, and has no analogue in the steady flow problem. It is naturally expected that, for the case with finite r_0, the intermediate layer of the thickness $O(E^{\frac{1}{4}})$ coexists together with Ekman layer also along the vertical wall.

4. Solution in the case with finite radius of the cylinder

Quantities in the inner inviscid region are expanded as follows:

$$q(r,\theta,z) = q^{(0)}(r,\theta,z)+E^{\frac{1}{4}}q^{(1)}(r,\theta,z)+E^{\frac{1}{2}}q^{(2)}(r,\theta,z)+\dots. \tag{23}$$

In the above expression, every order of approximation is assumed to be of the order of 1 with respect to its arguments. This is the scaling assumption by which the validity of our method of solution is examined. The same is the case for every other expansion.

In the neighbourhood of horizontal boundaries, quantities are expanded as follows:

$$
\begin{aligned}
q(r,\theta,z) = {} & q^{(0)}(r,\theta,z)+E^{\frac{1}{4}}q^{(1)}(r,\theta,z)+E^{\frac{1}{2}}q^{(2)}(r,\theta,z)+\dots \\
& +\hat{q}_j^{(0)}(r,\theta,\xi_j)+E^{\frac{1}{4}}\hat{q}_j^{(1)}(r,\theta,\xi_j)+E^{\frac{1}{2}}\hat{q}_j^{(2)}(r,\theta,\xi_j)+\dots \\
& +\hat{\hat{q}}_j^{(0)}(r,\theta,\eta_j)+E^{\frac{1}{4}}\hat{\hat{q}}_j^{(1)}(r,\theta,\eta_j)+E^{\frac{1}{2}}\hat{\hat{q}}_j^{(2)}(r,\theta,\eta_j)+\dots,
\end{aligned}
\tag{24}
$$

where

$$z = j+(-1)^j E^{\frac{1}{4}}\xi_j, \quad (j=0,1), \tag{25}$$

$$z = j+(-1)^j E^{\frac{1}{2}}\eta_j, \quad (j=0,1), \tag{26}$$

and quantities with hats and double hats are the corrections to those of the inviscid region corresponding, respectively, to the intermediate and to the Ekman layer. Since the boundary layers are restricted to the neighbourhood of the boundaries, these corrections must fade away at the edge of the respective layers.

Similar expansions are applied in the neighbourhood of the vertical boundary:

$$
\begin{aligned}
q(r,\theta,z) = {} & q^{(0)}(r,\theta,z)+E^{\frac{1}{4}}q^{(1)}(r,\theta,z)+E^{\frac{1}{2}}q^{(2)}(r,\theta,z)+\dots \\
& +\bar{q}^{(0)}(\alpha,\theta,z)+E^{\frac{1}{4}}\bar{q}^{(1)}(\alpha,\theta,z)+E^{\frac{1}{2}}\bar{q}^{(2)}(\alpha,\theta,z)+\dots \\
& +\bar{\bar{q}}^{(0)}(\beta,\theta,z)+E^{\frac{1}{4}}\bar{\bar{q}}^{(1)}(\beta,\theta,z)+E^{\frac{1}{2}}\bar{\bar{q}}^{(2)}(\beta,\theta,z)+\dots,
\end{aligned}
\tag{27}
$$

where

$$r = r_0 - E^{\frac{1}{4}}\alpha, \tag{28}$$

$$r = r_0 - E^{\frac{1}{2}}\beta. \tag{29}$$

Apart from the above boundary layers, there exist corner boundary layers where the horizontal and the vertical boundary layers merge to compose a complicated structure. This is the reason why the corner points are omitted from the following treatment. We believe that the property of this corner structure does not play an important role in the determination of the spin down procedure.

The substitution of (23) to (29) into (11) to (13) leads to the following zeroth-order equations.

In the inner inviscid region:

$$2\frac{\partial q_\theta^{(0)}}{\partial z}-\frac{\partial T^{(0)}}{\partial r} = 0, \tag{30}$$

$$\frac{\partial q_\theta^{(0)}}{\partial t}+2\frac{\partial \psi^{(0)}}{\partial z} = 0, \tag{31}$$

$$\sigma\frac{\partial T^{(0)}}{\partial t}-\frac{\sigma S}{r}\frac{\partial}{\partial r}(r\psi^{(0)}) = 0. \tag{32}$$

In the horizontal layers:

$$\left(\frac{\partial^2}{\partial \xi_j^2}-\sigma\frac{\partial}{\partial t}\right)\hat{T}_j^{(0)} = 0, \tag{33}$$

$$\frac{\partial^4 \hat{\hat{\psi}}_j^{(0)}}{\partial \eta_j^4}+2(-1)^j\frac{\partial \hat{\hat{q}}_{\theta j}^{(0)}}{\partial \eta_j} = 0, \tag{34}$$

$$\frac{\partial^2 \hat{\hat{q}}_{\theta j}^{(0)}}{\partial \eta_j^2}-2(-1)^j\frac{\partial \hat{\hat{\psi}}_j^{(0)}}{\partial \eta_j} = 0. \tag{35}$$

In the vertical layers:
$$\left(\frac{\partial^2}{\partial \alpha^2}-\frac{\partial}{\partial t}\right)\bar{q}_\theta^{(0)} = 0, \tag{36}$$

$$\frac{\partial^4 \overline{\overline{\psi}}^{(0)}}{\partial \beta^4}+\frac{\partial \overline{\overline{T}}^{(0)}}{\partial \beta} = 0, \tag{37}$$

$$\frac{\partial^2 \overline{\overline{T}}^{(0)}}{\partial \beta^2}-\sigma S\frac{\partial \overline{\overline{\psi}}^{(0)}}{\partial \beta} = 0, \tag{38}$$

where vanishing quantities are omitted for the sake of simplicity.

The zeroth-order boundary conditions are as follows:

On the horizontal boundary:
$$\psi^{(0)}+\hat{\hat{\psi}}_j^{(0)} = 0, \tag{39}$$

$$q_\theta^{(0)}+\hat{\hat{q}}_{\theta j}^{(0)} = \omega r, \tag{40}$$

$$\frac{\partial \hat{\hat{\psi}}_j^{(0)}}{\partial \eta_j} = 0, \tag{41}$$

$$\frac{\partial \hat{T}_j^{(0)}}{\partial \xi_j} = 0, \tag{42}$$

for $z = j\ (j=0,1), \quad 0 \leqslant r < r_0$.

On the vertical boundary:
$$\psi^{(0)}+\overline{\overline{\psi}}^{(0)} = 0, \tag{43}$$

$$q_\theta^{(0)}+\bar{q}_\theta^{(0)} = \omega r_0, \tag{44}$$

$$\frac{\partial \overline{\overline{\psi}}^{(0)}}{\partial \beta} = 0, \tag{45}$$

$$\frac{\partial \overline{\overline{T}}^{(0)}}{\partial \beta} = 0. \tag{46}$$

for $0 < z < 1, \quad r = r_0$.

The initial conditions are as follows:

all zeroth-order quantities vanish for $t \leqslant 0$. (47)

The solution of (37) and (38) satisfying (45) and (46) is trivial:
$$\overline{\overline{\psi}}^{(0)} = \overline{\overline{T}}^{(0)} = 0. \tag{48}$$

This leads to the following by (43):
$$\psi^{(0)} = 0 \quad \text{for} \quad 0 < z < 1, \quad r = r_0. \tag{49}$$

This, in turn, leads to the following using (31):
$$\frac{\partial q_\theta^{(0)}}{\partial t} = 0 \quad \text{for} \quad 0 < z < 1, \quad r = r_0. \tag{50}$$

The inviscid peripheral velocity at the edge of the vertical boundary layer is preserved in a spin down time scale, as is stated correctly by Pedlosky (1967). This does not, however, mean that the meridional current is completely quenched, but that the meridional current does close itself by crawling along the edge of the vertical boundary layer. In effect, as is shown presently, we can obtain the solution satisfying the necessary initial and boundary conditions including (49). The neglect of this crucial point is the reason why Pedlosky came to his erroneous conclusion.

The first-order equations are summarized as follows:

$$\left(\frac{\partial^2}{\partial \xi_j^2} - \sigma \frac{\partial}{\partial t}\right) \hat{T}_j^{(1)} = 0, \tag{51}$$

$$\frac{\partial \bar{T}^{(1)}}{\partial \alpha} = -2\frac{\partial \bar{q}_\theta^{(0)}}{\partial z}, \tag{52}$$

$$\left(\frac{\partial^2}{\partial \alpha^2} - \frac{\partial}{\partial t}\right) \bar{q}_\theta^{(1)} = \frac{1}{r_0}\frac{\partial \bar{q}_\theta^{(0)}}{\partial \alpha}, \tag{53}$$

$$\frac{\partial \hat{\hat{T}}_j^{(2)}}{\partial \eta_j} + \frac{\partial \hat{T}_j^{(1)}}{\partial \xi_j} + (-1)^j \frac{\partial T^{(0)}}{\partial z} = 0 \quad \text{for} \quad z = j\ (j = 0, 1), \quad 0 \leqslant r < r_0, \tag{54}$$

$$\bar{q}_\theta^{(1)} = 0 \quad \text{for} \quad 0 < z < 1, \quad r = r_0, \tag{55}$$

where vanishing quantities are again omitted for the sake of simplicity. It is interesting to note that the second-order quantity appears in the description of the first-order ones via the boundary conditions (54). The equation for this second-order quantity is as follows:

$$\frac{\partial^2 \hat{\hat{T}}_j^{(2)}}{\partial \eta_j^2} = -\frac{\sigma S}{r}\frac{\partial}{\partial r}(r \hat{\hat{\psi}}_j^{(0)}). \tag{56}$$

Once $\hat{\hat{\psi}}_j^{(0)}$ is known beforehand, as certainly is the case, $\hat{\hat{T}}_j^{(2)}$ is uniquely determined by its boundary-layer character. Thus, our first-order equation can be solved without any trouble. The examination of higher order approximations elucidates that this property of self-containedness is inherited by every order of approximation and assures the validity of our method of solution.

The second-order equations for the vertical wall Ekman layer are as follows:

$$\frac{\partial^4 \bar{\bar{\psi}}^{(2)}}{\partial \beta^4} + \frac{\partial \bar{\bar{T}}^{(2)}}{\partial \beta} = 0, \tag{57}$$

$$\frac{\partial^2 \bar{\bar{T}}^{(2)}}{\partial \beta^2} - \sigma S \frac{\partial \bar{\bar{\psi}}^{(2)}}{\partial \beta} = 0, \tag{58}$$

$$\left.\begin{aligned} \frac{\partial \bar{\bar{\psi}}^{(2)}}{\partial \beta} &= \frac{\partial \psi^{(0)}}{\partial r}, \\ \frac{\partial \bar{\bar{T}}^{(2)}}{\partial \beta} &= \frac{\partial T^{(0)}}{\partial r} - \frac{\partial \bar{T}^{(1)}}{\partial \alpha} \end{aligned}\right\} \quad \text{for} \quad 0 < z < 1, \quad r = r_0. \tag{59}$$

Thus, the second-order vertical wall Ekman layer is passively induced by the zeroth-order inviscid flow. Holton's (1965) statement that the vertical wall boundary layer has only a passive role is quite right, provided that his solution corresponds to the present case with thermally insulated boundary. Unfortunately, however, his solution neither corresponds to the present case nor to the case with given boundary temperature (as is seen by the comparison between equation (22) of his paper and equation (70) in the following and equation (6.29) of Pedlosky's paper).

By the introduction of the Laplace transform, it is shown that (34) and (35) have the following solutions under the boundary conditions (39) and (40):

$$\hat{\hat{Q}}_{\theta j}^{(0)} = (-1)^{1+j}(1-i)A_j e^{-(1+i)\eta_j} + (-1)^{1+j}(1+i)B_j e^{-(1-i)\eta_j}, \tag{60}$$

$$\hat{\hat{\Psi}}_j^{(0)} = A_j e^{-(1+i)\eta_j} + B_j e^{-(1-i)\eta_j}, \tag{61}$$

$$A_j = \frac{i}{2}\left\{(1+i)\Psi_j^{(0)} + (-1)^{1+j}\left(\frac{r\omega}{\tau} - Q_{\theta j}^{(0)}\right)\right\}, \tag{62}$$

$$B_j = \frac{i}{2}\left\{-(1-i)\Psi_j^{(0)} + (-1)^{j}\left(\frac{r\omega}{\tau} - Q_{\theta j}^{(0)}\right)\right\}, \tag{63}$$

where capital letters are the transformed quantities of corresponding ones designated by small letters and suffix j refers to the top and bottom according to its value (1 and 0). It is noted that the initial condition (47) is used in the transformation of the basic equations. The imposition of the boundary condition (41) leads to the following boundary conditions to be satisfied by the inviscid flow:

$$2\tau\Psi^{(0)} + 2(-1)^{1+j}\frac{\partial\Psi}{\partial z} = (-1)^j r\omega \quad \text{for} \quad z = j\,(j=0,1), \quad 0 \leqslant r < r_0. \tag{64}$$

The transformed versions of (30) to (32) are as follows:

$$S\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\Psi^{(0)})\right\} + 4\frac{\partial^2\Psi^{(0)}}{\partial z^2} = 0, \tag{65}$$

$$Q_\theta^{(0)} = -\frac{2}{\tau}\frac{\partial\Psi^{(0)}}{\partial z}, \tag{66}$$

$$\mathscr{T}^{(0)} = \frac{S}{\tau r}\frac{\partial}{\partial r}(r\Psi^{(0)}). \tag{67}$$

Our problem is reduced to the solution of (65) under the boundary conditions (49) and (64).

The above boundary value problem is easily solved to give the following:

$$\Psi^{(0)} = r_0\omega\sum_{n=1}^{\infty}\frac{\sinh\{\Omega_n(1-z)\} - \sinh(\Omega_n z)}{\omega_n J_2(\omega_n)(\tau\sinh\Omega_n + \Omega_n\cosh\Omega_n + \Omega_n)} J_1\left(\omega_n\frac{r}{r_0}\right), \tag{68}$$

where

$$\Omega_n = \frac{S^{\frac{1}{2}}\omega_n}{2r_0}, \tag{69}$$

and $J_n(x)$ and ω_n are the Bessel function of the first kind of order n and zeros of $J_1(x)$, respectively.

The substitution of (68) into (66) and (67), and the inversion of the transformation leads to the following:

$$\frac{q_\theta^{(0)}}{2r_0\omega} = \sum_{n=1}^{\infty}\left\{1 - \exp\left(-\frac{\Omega_n(\cosh\Omega_n + 1)}{\sinh\Omega_n}t\right)\right\}\frac{\cosh(\Omega_n z) + \cosh\{\Omega_n(1-z)\}}{\omega_n J_2(\omega_n)(1+\cosh\Omega_n)} J_1\left(\omega_n\frac{r}{r_0}\right), \tag{70}$$

$$\frac{T^{(0)}}{S\omega} = \sum_{n=1}^{\infty}\left\{1 - \exp\left(-\frac{\Omega_n(\cosh\Omega_n + 1)}{\sinh\Omega_n}t\right)\right\}\frac{\sinh\{\Omega_n(1-z)\} - \sinh(\Omega_n z)}{\Omega_n J_2(\omega_n)(1+\cosh\Omega_n)} J_0\left(\omega_n\frac{r}{r_0}\right). \tag{71}$$

It is easily shown that the zeroth-order approximations (70) and (71) are finite in the inviscid region: $0 \leqslant r < r_0$, $0 < z < 1$. The similar result is true for the boundary layer solutions. Thus, our zeroth-order approximation is self-consistent. The same is proved to be true by examination of the higher order approximations.

5. Results and discussion

The velocity and the temperature in the inviscid region depends on only one parameter $S^{\frac{1}{2}}/r_0$ except for factors of proportionality. The case with infinitely large radius is equivalent to that with vanishing effect of stratification. Thus, our solution certainly includes the similar solution in § 3 as its limiting case. The asymptotic velocity on the edge of the horizontal boundary layers is shown to be that of the rigid rotation by (70) and the following:

$$\frac{r}{r_0} = \sum_{n=1}^{\infty} \frac{2}{\omega_n J_2(\omega_n)} J_1\left(\omega_n \frac{r}{r_0}\right). \tag{72}$$

On the other hand, the velocity on the edge of the vertical wall boundary layers is that of the unperturbed state throughout the time scale of the spin down time. This is a good manifestation of the quasi-steady nature of the asymptotic state. This situation becomes clearer by the examination of the asymptotic amount of total angular momentum transferred into the inviscid region via the meridional current:

$$\frac{J}{J_r} = 16 \sum_{n=1}^{\infty} \frac{\sinh \Omega_n}{\omega_n^2 \Omega_n(\cosh \Omega_n + 1)}, \tag{73}$$

where

$$J = 2\pi \int_0^L \int_0^{Lr_0} \bar{r}^2 \bar{\rho} \bar{q}_\theta \, d\bar{r} \, d\bar{z}, \tag{74}$$

and suffix r refers to the new state of rigid rotation. By the integration of

$$f(\zeta) = \frac{S \sinh \zeta \, J_0(2r_0 \zeta / S^{\frac{1}{2}})}{(2r_0 \zeta)^2 \zeta(\cosh \zeta + 1) J_1(2r_0 \zeta/S^{\frac{1}{2}})}, \tag{75}$$

along suitable path with respect to ζ and the application of the theory of residues, the above is rewritten as follows:

$$\frac{J}{J_r} = 1 + \frac{1}{6}\frac{S}{r_0^2} - \frac{16 S^{\frac{1}{2}}}{r_0 \pi^3} \sum_{n=0}^{\infty} \frac{I_0(2r_0\pi(2n+1)/S^{\frac{1}{2}})}{(2n+1)^3 I_1(2r_0\pi(2n+1)/S^{\frac{1}{2}})}, \tag{76}$$

where $I_n(x)$ is the modified Bessel function of the first kind of order n. Equation (76) assures the continuous approach to the homogeneous-fluid limit, while (73) shows the smooth fading away of the spin down procedure in the limit of very strong stratification. The calculated value of J/J_r is tabulated in table 1. The third column of the above table is the value for the case with given boundary temperature. The amount of the transferred angular momentum is much less in the present case. We can say that the meridional current is more readily interrupted in the case with thermally insulated boundary. However, this does not mean the complete quenching of the meridional current. The meridional current does exist within the time scale of the homogeneous-fluid spin down time. Only, its effect is weakened by the effect of the stable stratification. We stand completely on the side of Holton in this respect. The secondary meridional current induced by the Ekman layers spins the fluid down to a quasi-steady state within the time scale of the homogeneous-fluid spin down time. However, unlike the homogeneous case, the quasi-steady state is not one of solid body rotation. Final approach to the state of rigid rotation is achieved via viscous diffusion in the time scale of the usual diffusion time.

	J/J_r	
$S^{\frac{1}{2}}/r_0$	Thermally insulated wall	Boundary temperature given
0	1	1
0·25	0·8720	0·9971
0·5	0·7592	0·9894
0·75	0·6609	0·9782
1	0·5765	0·9648
1·25	0·5051	0·9504
1·5	0·4458	0·9363

TABLE 1.

The average temperature on each horizontal plane in the inviscid region is equal to that of the unperturbed state:

$$\int_0^1 T^{(0)} r\, dr = 0. \tag{77}$$

This corresponds to the situation that thermal energy is not transferred across the boundary.

The implication of our results on Dicke's solar spin down controversy is clear. At least, he must abandon Pedlosky's erroneous conclusion as the support of his hypothesis. He can, of course, rely on the non-uniformity of the quasi-steady state and also to maintain that our results do not apply on the basis of the limitation of Boussinesq approximation. However, this is not a problem of quality but of quantity. The best way to resolve the solar spin down controversy is to investigate the unsteady rotational motion of the model solar interior.

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Note added in proof

Thermal part of (17) does not exactly correspond to the thermal insulation of the horizontal wall. However, this causes no limitations on applicability of our results. As is shown in (33), (42), (51), (54) and (56), the temperature distribution in the horizontal boundary layers does not affect those of the other physical quantities. In effect, the derivation of (64) is completely independent of the thermal condition on the horizontal wall. Therefore, our results apply not only to the case described by (17) but also to any other cases as far as the thermal condition on the horizontal wall is concerned; for example, to the case with unperturbed temperature of the horizontal wall as well as the case with thermally insulated horizontal wall.